The Theory of Interest

Interest is money paid by a bank or other financial institution to an investor or depositor in exchange for the use of the depositor's money.
Amount of interest is (usually) a fraction (called the **interest rate**) of the initial amount deposited called the **principal amount**.
Notation:

- r : interest rate per unit time
- C: principal amount
- FV: amount due (account balance) – Future Value
- t: time

These quantities are related through the equation:

$$FV = C*(1 + r*t).$$

Once credited to the investor, the interest may be kept by the investor, and may earn interest itself. If interest is credited once per year, then after *t* years the amount due is

$$FV = C * (1 + r)^t$$

If a portion of the interest is credited after a fraction of a year, then the interest is said to be **compounded**. If there are *n* **compounding periods** per year, then in *t* years the amount due is

$$FV = C * (1 + \frac{r}{n})^{n*t}$$

Suppose an account earns 5.75% annually compounded monthly. If the principal amount is $3104 what is the amount due after three and one-half years?
Solution:

$$FV = C * (1 + \frac{r}{n})^{n*t} = \$3,104 * \left(1 + \frac{5.75\%}{12}\right)^{3.5*12} = \$3,794.15$$

Suppose an account earns 5.75% annual simple interest. If the principal amount is $3104 what is the amount due after three and one-half years?

$$FV = C * (1 + r * t) = \$3,104 * (1 + 5.75\% * 3.5) = \$3,728.68$$

The annual interest rate equivalent to a given compound interest rate is called the **effective interest rate**.

$$r_e = \left(1 + \frac{r}{n}\right)^n - 1$$

Suppose an account earns 5.75% annually compounded monthly. What is the effective interest rate?

$$r_e = \left(1 + \frac{r}{n}\right)^n - 1 = \left(1 + \frac{5.75\%}{12}\right)^{12} - 1 = 5.90\%$$

What happens as we increase the frequency of compounding? The amount due for c**ontinuously compounded interest or effective interest rate** is

$$FV = C * e^{r*t}$$

Suppose $3585 is deposited in an account which pays interest at an annual rate of 6.15% compounded continuously.

- Find the amount due after two and one half years.
- Find the equivalent annual effective simple interest rate.

Amount due is

$$FV = C * e^{r*t} = \$3,585 * e^{6.15\%*2.5} = \$4,180.82$$

Effective interest rate is

$$r_e = e^r - 1 = e^{6.15\%} - 1 = 6.34\%$$

How do we rationally compare amounts of money paid at different times in an interest-bearing environment?
The **present value** of *a future amount C*, an amount due *t* years from now subject to an interest rate *r* is the principal amount *P* which must to invested now so that *t* years from now the accumulated principal and interest total *C*.

$$PV = \frac{C}{\left(1+\frac{r}{n}\right)^{n*t}} = C * \left(1 + \frac{r}{n}\right)^{-n*t}$$ with discrete compounding

$$PV = \frac{C}{e^{r*t}} = C * e^{-r*t}$$ with continuous compounding.

Suppose an investor will receive payments at the end of the next six years in the amounts shown in the table.
Year 1 2 3 4 5 6
Payment 465 233 632 365 334 248

Year	1	2	3	4	5	6
Payment	465	233	632	365	334	248

If the interest rate is 3.99% compounded monthly, what is the present value of the investments?

$$PV = \frac{465}{\left(1+\frac{3.99\%}{12}\right)^{12*1}} + \frac{233}{\left(1+\frac{3.99\%}{12}\right)^{12*2}} + \frac{632}{\left(1+\frac{3.99\%}{12}\right)^{12*3}} + \frac{365}{\left(1+\frac{3.99\%}{12}\right)^{12*4}} + \frac{334}{\left(1+\frac{3.99\%}{12}\right)^{12*5}} + \frac{248}{\left(1+\frac{3.99\%}{12}\right)^{12*6}} = \$2,003$$

Suppose a loan of amount *P* will be paid back discretely (*n* times per year) over *t* years. The unpaid portion of the loan is charged interest at annual rate *r* compounded *n* times per year. What is the discrete payment *c*?
Hint: the present value of all the payments should equal the amount borrowed.
If the first payment must be made at the end of the first compounding period, then the present value of all the payments is

$$PV = c * \frac{\left(1+\frac{r}{n}\right)^{n*t} - 1}{\frac{r}{n} * \left(1+\frac{r}{n}\right)^{n*t}} => c = PV * \frac{\frac{r}{n} * \left(1+\frac{r}{n}\right)^{n*t}}{\left(1+\frac{r}{n}\right)^{n*t} - 1}$$

If a person borrows $25,000 for five years at an interest rate of 4.99% compounded monthly and makes equal monthly payments, what is the monthly payment?

$$c = PV * \frac{\frac{r}{n} * \left(1+\frac{r}{n}\right)^{n*t}}{\left(1+\frac{r}{n}\right)^{n*t} - 1} = \$25,000 * \frac{\frac{4.99\%}{12} * \left(1+\frac{4.99\%}{12}\right)^{12*5}}{\left(1+\frac{4.99\%}{12}\right)^{12*5} - 1}$$

=$471.67

Suppose a person is 25 years of age now and plans to retire at age 65. For the next 40 years they plan to invest a portion of their monthly income in securities which earn interest at the rate of 10% compounded monthly. After retirement the person plans on receiving a monthly payment (an annuity) in the absolute amount of $1500 for 30 years. How much should be set aside monthly for retirement?
The present value of all funds invested for retirement should equal the present value of all funds taken out during retirement.

$$PV = c * \frac{\left(1+\frac{r}{n}\right)^{n*t} - 1}{\frac{r}{n} * \left(1+\frac{r}{n}\right)^{n*t}} = \$1,500 * \frac{\left(1+\frac{10\%}{12}\right)^{12*30} - 1}{\frac{10\%}{12} * \left(1+\frac{10\%}{12}\right)^{12*30}} =$$

An increase in the amount of money in circulation without a commensurate increase in the amount of available goods is a condition known as inflation. Thus relative to the supply of goods, the value of the currency is decreased.
How does inflation (measured at an annual rate i) affect the value of deposits earning interest?

- Suppose at the current time one unit of currency will purchase one unit of goods.
- Invested in savings, that one unit of currency has a future value (in one year) of 1 + r.
- In one year the unit of goods will require 1 + i units of currency for purchase.

- The difference (1 + r) − (1 + i) = r − i will be the real rate of growth in the unit of currency invested now.
- This return on saving will not be earned until one year from now. The present value of r − i under inflation rate i is

$$r_r = \frac{1+r}{1+i}$$

Suppose a person is 25 years of age now and plans to retire at age 65. For the next 40 years they plan to invest a portion of their monthly income in securities which earn interest at the rate of 10% compounded monthly. After retirement the person plans on receiving a monthly payment (an annuity) in the absolute amount of $1500 for 30 years. How much should be set aside monthly for retirement if the annual inflation rate is 3%?

The inflation adjusted return on saving is

$$r_r = \frac{1+r}{1+i} = \frac{1+10\%}{1+3\%} = 6.80\%$$

Using this value in place of r in the previous example we have

$$PV = c * \frac{\left(1+\frac{r}{n}\right)^{n*t} - 1}{\frac{r}{n} * \left(1+\frac{r}{n}\right)^{n*t}} = \$1{,}500 * \frac{\left(1+\frac{6.80\%}{12}\right)^{12*30} - 1}{\frac{6.80\%}{12} * \left(1+\frac{6.80\%}{12}\right)^{12*30}} =$$

Suppose a person takes out an mortgage loan in the amount of PV and will make n equal monthly payments of amount x where the annual interest rate is r compounded monthly.

- Express c as a function of PV, r, and n.
- After the jth month, how much of the original amount borrowed remains?
- How much of the jth payment goes to interest and how much goes to pay down the amount borrowed?

The sum of the present values of all the payments must equal the amount loaned.